THE LAPLACE-POISSON MIXED EQUATION.

By K. P. Williams.

In a recent number of this Journal * Borden discussed the Laplace-Poisson Mixed Equation

(1) $$f'(x+1) + p(x)f'(x) + q(x)f(x+1) + r(x)f(x) = 0.$$

He obtains two invariants that form a fundamental system for the equation. These invariants are

$$I(x) = \frac{r(x)}{p(x)} - q(x) - \frac{p'(x)}{p(x)},$$
$$J(x) = \frac{r(x)}{p(x)} - q(x-1).$$

If either invariant is zero, the equation takes a simple form. For instance, if $I(x) \equiv 0$, i.e., if

(2) $$r(x) \equiv p'(x) + p(x)q(x),$$

the equation (1) reduces to

(3) $$f(x+1) + p(x)f(x) \equiv Ce^{-\int q(x)dx},$$

where C is an arbitrary constant. This is a linear non-homogeneous difference equation of the first order. Borden obtains a solution of it by means of the symbolic solution $\Sigma G(x)$ of the equation

$$F(x+1) - F(x) = G(x),$$

where $G(x)$ is a known function.

Solutions obtained in this way are purely formal, and may have no real significance. Borden assumes at the outset of his paper that $p(x)$, $q(x)$, and $r(x)$ are analytic functions, but he nowhere makes an investigation to determine whether this hypothesis is sufficient to bestow any validity upon his results.

It is the purpose of this paper to investigate in certain cases the analytic character of the solutions Borden obtains. To do this it is necessary to make use of the existence theorems for linear difference equations. We shall state here the one that is sufficient for several cases that arise.

* R. F. Borden, "On the Laplace-Poisson Mixed Equation," American Journal of Mathematics, Vol. XLII, 1920, pp. 257–277.

THEOREM A.* *Consider the linear non-homogeneous difference equation*

$$g(x+1) + a(x)g(x) = b(x), \tag{4}$$

where $a(x)$ and $b(x)$ are rational functions, so that

$$a(x) = x^{\mu}\left(a_0 + \frac{a_1}{x} + \cdots\right),$$

$$b(x) = x^{\nu}\left(b_0 + \frac{b_1}{x} + \cdots\right), \qquad |x| > R.$$

There exists a series $g(x)$, of the form

$$\bar{g}(x) = x^{\nu-\mu}\left(g_0 + \frac{g_1}{x} + \cdots\right), \qquad \text{if} \qquad \mu > 0,$$

$$\bar{g}(x) = x^{\nu}\left(g_0 + \frac{g_1}{x} + \cdots\right), \qquad \text{if} \qquad \mu \leqq 0,$$

which formally satisfies (4), *but which in general diverges.*

There are two solutions $g_1(x)$ and $g_2(x)$ with the following properties. The function $g_1(x)$ is analytic save at zeros of $a(x)$, poles of $b(x)$, and points congruent (*mod* 1) *to these points on the left; and it is asymptotic to the formal series $\bar{g}(x)$ in the right half plane. The function $g_2(x)$ is analytic save for poles at the poles of $a(x)$, $b(x)$, and the points congruent to these points on the right; and it is asymptotic to $\bar{g}(x)$ in the left half plane.*†

From the relation (2) it is possible to determine any one of the three functions p, q, r in terms of the other two. We shall assume that p and q have the form of rational functions at infinity, so that

$$\begin{aligned} p(x) &= x^{m}\left(p_0 + \frac{p_1}{x} + \cdots\right), \\ q(x) &= x^{n}\left(q_0 + \frac{q_1}{x} + \cdots\right), \qquad |x| > R. \end{aligned} \tag{5}$$

There are four cases to consider, according as $n < -1$, $n = -1$, $n = 0$, $n > 0$.

* K. P. Williams, "The Solutions of Non-Homogeneous Linear Difference Equations and their Asymptotic Form," *Transactions of the American Math. Soc.*, Vol. XIV (1913), pp. 209–240.

† In order to be assured of solutions and to know their asymptotic forms, it is sufficient to assume that $a(x)$ and $b(x)$ have the form of rational functions at infinity. In that case, nothing can be said of the nature or situation of the singularities of the solutions in the finite part of the plane.

§ 1.

Case 1. Let $n < -1$; then we have

$$e^{-\int q(x)dx} = \left(1 + \frac{}{x^{-n-1}} + \frac{}{x^{-n-2}} + \cdots\right).^{*}$$

It is seen that equation (3) is now in the form (4), with $\mu = m$, $\nu = 0$. It follows from Theorem A that equation (3), and therefore equation (1), has two solutions, analytic in general, and these solutions are asymptotic in the right and left half planes, respectively, to a series

$$\bar{f}(x) = x^{-m}\left(f_0 + \frac{f_1}{x} + \cdots\right), \qquad \text{if} \qquad m > 0,$$

$$\bar{f}(x) = f_0 + \frac{f_1}{x} + \cdots, \qquad \text{if} \qquad m \leqq 0,$$

which can be determined by direct substitution in (3).

The formal solution $\bar{f}(x)$ can, however, be directly obtained from (1). The calculation will be made only for the case $m > 0$.

When we substitute in (2) the values of $p(x)$ and $q(x)$ given in (5), we obtain

$$r(x) = x^{m-1}\left(mp_0 + \frac{}{x} + \cdots\right),$$

the highest power of x being the $(m-1)$th, since the assumption $n < -1$ gives $m - 1 > m + n$. The equation (1) then takes the form

$$f'(x+1) + x^m\left(p_0 + \frac{p_1}{x} + \cdots\right)f'(x)$$
$$+ x^n\left(q_0 + \frac{q_1}{x} + \cdots\right)f(x+1) + x^{m-1}\left(mp_0 + \frac{}{x} + \cdots\right)f(x) = 0.$$

Assume

$$f(x) = x^{-m}\left(f_0 + \frac{f_1}{x} + \cdots\right),$$

and substitute, giving

$$-\frac{1}{x^{m+1}}(mf_0 + \cdots) - \frac{1}{x}(p_0 + \cdots)(mf_0 + \cdots)$$
$$+ \frac{1}{x^{m-n}}(q_0 + \cdots)(f_0 + \cdots) + \frac{1}{x}(mp_0 + \cdots)(f_0 + \cdots) = 0,$$

where we have omitted only powers of $1/x$. Remembering that $m > 0$, $n < -1$, we see that $f_0, f_1, f_2, \cdots$ can be determined step by step, the quantity f_0 being arbitrary.†

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* We shall omit writing coefficients in series in $1/x$ where the coefficients can be determined in terms of given quantities, and their explicit form is of no concern.

† If $m \leqq 0$ it is found that f_0 is arbitrary, while some of the next coefficients, $f_1, f_2, \ldots$, are zero, the exact number having that value depending on m and n.

THEOREM 1. *Assume the coefficients p, q, and r satisfy the following hypotheses*:

(1) $$r(x) \equiv p'(x) + p(x)q(x).$$

(2) *$p(x)$ has a pole of order m at infinity ($m = 0$, if $p(x)$ is analytic at infinity).*

(3) *$q(x)$ is analytic at infinity, and has a zero there of at least the second order.*

Then there is a formal solution $\bar{f}(x)$ of equation (1) *of the form*

$$\bar{f}(x) = x^{-m}\left(f_0 + \frac{f_1}{x} + \cdots\right),$$

which can be determined by direct substitution (f_0 is arbitrary). There are two solutions of (1), *namely $f_1(x)$ and $f_2(x)$, analytic in general in the finite plane. Furthermore, $f_1(x)$ is asymptotic to $\bar{f}(x)$ in the right half plane, and $f_2(x)$ is asymptotic to $\bar{f}(x)$ in the left half plane.*

§ 2.

Case 2. Let $n = -1$. In this case

$$e^{-\int q(x)dx} = x^{-q_0}\left(1 + \frac{}{x} + \frac{}{x^2} + \cdots\right).$$

The equation (3) is therefore of the form (4) with $\nu = -q_0$.*

By application of Theorem A we deduce

THEOREM 2. *Let p, q, and r satisfy hypotheses* (1) *and* (2) *of Theorem* 1, *and in addition suppose*

(3) *$q(x)$ is analytic at infinity, with a zero of the first order, and* $\lim_{x=\infty} xf(x) = q_0$.

Then there is a formal solution $\bar{f}(x)$ of (1) *of the form*

$$\bar{f}(x) = x^{-q_0-m}\left(f_0 + \frac{f_1}{x} + \cdots\right),$$

which can be determined by substitution. There are two solutions $f_1(x)$ and $f_2(x)$ with properties similar to those described in Theorem 1.

§ 3.

Case 3. Let $n = 0$. In this case

$$e^{-\int q(x)dx} = e^{-q_0x}x^{-q_1}\left(1 + \frac{}{x} + \cdots\right),$$

* It is not necessary that ν be an integer in (3). In fact the substitution

$$g(x) = x^{\nu}G(x)$$

will reduce (4) to a normal form with $\nu = 0$, and in which the expansion of $a(x)$ still begins with x^{μ}.

so that equation (3) is not of the form (4). It can, however, be reduced to that form by the substitution

$$f(x) = e^{-q_0 x} g(x).$$

The new equation will be

$$g(x+1) + \frac{1}{e^{q_0}} p(x) g(x) = \frac{x^{-q_1}}{e^{q_0}}\left(1 + \frac{}{x} + \cdots\right).$$

The application of Theorem A then gives

THEOREM 3. *Let p, q, r satisfy hypotheses* (1) *and* (2) *of Theorem* 1, *and in addition assume*

(3) *$q(x)$ is analytic at infinity, with*

$$\lim_{x=\infty} q(x) = q_0, \qquad \lim_{x=\infty} x(q(x) - q_0) = q_1.$$

Then there is a formal solution $\bar{f}(x)$ of (1) *of the form*

$$\bar{f}(x) = e^{-q_0 x} x^{-q_1 - m}\left(f_0 + \frac{f_1}{x} + \cdots\right).$$

Solutions $f_1(x)$ and $f_2(x)$ exist, and have properties similar to those described before.

§ 4.

Case 4. Let $n > 1$. We can now write

$$e^{-\int q(x)dx} = e^{-q^{(1)}(x)} \cdot q_2(x),$$

where

$$q^{(1)}(x) = q_0 \frac{x^{n+1}}{n+1} + \cdots + q_n x,$$

$$q^{(2)}(x) = x^{-q_{n+1}}\left(1 + \frac{q_{n+2}}{x} + \frac{}{x^2} + \cdots\right),$$

the quantities $q_0, q_1, \cdots$ having the significance given in (5). The equation (3) accordingly takes a form to which Theorem A is not applicable. A direct examination must therefore be made in order to determine whether a solution exists.

Consider the associated homogeneous equation

$$g(x+1) + p(x) g(x) = 0.$$

There is a formal solution

$$\bar{g}(x) = x^{mx}(-p_0 e^{-m})^x x^r\left(\bar{g}_0 + \frac{\bar{g}_1}{x} + \cdots\right), \tag{6}$$

*which in general diverges. There are two solutions $g_1(x)$ and $g_2(x)$, analytic in general in the finite plane, and asymptotic to $\bar{g}(x)$, in the right and left half planes, respectively.**

* G. D. Birkhoff, "General Theory of Linear Difference Equations," *Transactions of the American Math. Soc.*, Vol. 12 (1911), pp. 243–284.

Let $g(x)$ represent one (which one to be specified later) of these solutions, and put

(7) $$f(x) = \omega(x)g(x),$$

We then have, upon substituting in (3), the equation

(8) $$\omega(x+1) - \omega(x) = C\frac{e^{-q^{(1)}(x)}\cdot q^{(2)}(x)}{g(x+1)} = G(x).$$

The discussion of this equation falls into various cases.

Case 4a. Suppose $q_0 > 0$. Let $g(x)$ be $g_1(x)$. Then the series

(9) $$\omega(x) = -G(x) - G(x+1) - \cdots$$

is uniformly convergent if the real part of x is positive and sufficiently large; and the series is a solution of (8). The proof in all its details will not be given, but is based upon the following considerations.* If s is sufficiently large, the sth term in (9) can be written, on account of the fact that $g_1(x)$ is asymptotic in the right half plane to $\bar{g}(x)$, ($g(\bar{x})$ having the form given in (6)),

$$\frac{e^{-q^{(1)}(x+s-1)}}{(x+s)^{m(x+s)}(-p_0e^{-m})^{x+s}(x+s)^{r+q_{n+1}}}M(x, s),$$

where $M(x, s)$ is bounded. Now the dominant part of the numerator is, for large s,

$$e^{-\frac{q_0}{n+1}s^{n+1}}$$

and the dominant part of the denominator is $s^{ms+r+q_{n+1}}\cdot e^{-ms}$. The dominant part of the expression is therefore

$$e^{-\frac{q_0}{n+1}s^{n+1}+ms-(ms+r+q_{n+1})\log s}.$$

It is seen that this is the term of a rapidly convergent series, since q_0 is positive. This is true irrespective of the sign of m, for s^{n+1} dominates $s \log s$, for s large, if $n > 0$.

Case 4b. Suppose $q_0 < 0$, and let n be odd. Let $g(x)$ be $g_2(x)$. Then the series

(10) $$\omega(x) = G(x-1) + G(x-2) + \cdots$$

is uniformly convergent for the real part of x negative and sufficiently large; and it is a solution of (8). The proof is effected as before, making use of the fact that $g_2(x)$ is asymptotic to $\bar{g}(x)$ in the left half plane.

Case 4c. Suppose $q_0 < 0$, and let n be even. Neither a series of form (9) nor (10) will converge in this case, with $g(x)$ representing either $g_1(x)$ or $g_2(x)$. This is true because s^{n+1} dominates $s \log s$.

A solution of (8) can be obtained by means of a contour integral due

* Williams, loc. cit., § 3.

to Guichard.* The integral

$$\omega(x) = \int \frac{G(t)dt}{e^{2\pi i (t-x)} - 1}$$

satisfies (8), when the path of integration passes between $x - 1$ and x, with the point x on the right, provided the path extends to infinity in such a way that the integral converges.

In order to determine a choice of the path of integration we shall examine the nature of

$$G(t) = \frac{Ce^{-q^{(1)}(t)} \cdot q^{(2)}(t)}{g(t+1)} = - \frac{Ce^{-q^{(1)}(t)} \cdot q^{(2)}(t)}{p(t)g(t)} \,.$$

It is obvious that we can neglect $q^{(2)}(t)$ and $p(t)$, if we change the exponent r in the asymptotic form of $\bar{g}(x)$ to $r' = r + m + q_{n+1}$. (This change has the effect of making $q^{(2)}(t)$ and $p(t)$ approach constants as $t = \infty$.)

Let $g(t)$ be $g_1(t)$; then the dominating part in the denominator of $|G(t)|$ is, for t large in the right half plane,

$$|t^{mt}(-\,p_0 e^{-m})^t t^{r'}|.$$

If we put $t = u + iv$, and also $t = \tau e^{i\varphi}$ this becomes

$$e^{(mu+r')\log\tau-(m-\log-p_0)u-m\varphi v} = e^{(\tau m\cos\varphi+r')\log\tau-[(m-\log-p_0)\cos\varphi+m\varphi\sin\varphi]\tau}.$$

Now choose the path of integration so as to make it coincident with the lines

$$\varphi = \pm \frac{\pi}{n+1}$$

at a sufficient distance from $t = 0$. Then $t = \tau e^{i\frac{\pi}{n+1}}$ so that $t^{n+1} = -\,\tau^{n+1}$. It follows that the dominant part of the real portion of the polynomial $q^{(1)}(t)$ will be, for τ sufficiently great, the positive quantity

$$\frac{|q_0|\tau^{n+1}}{n+1} \,.$$

Along the contour considered it follows that

$$|G(t)| < M(\tau)e^{-\frac{|q_0|}{n+1}\tau^{n\ 1}-(\tau m\cos\varphi+r')\log\tau-[(m-\log-p_0)\cos\varphi+m\varphi\sin\varphi]\tau},$$

where $M(\tau)$ is bounded. It is seen that $G(t)$ will approach zero more rapidly than $e^{-\tau}$ as τ approaches infinity, and this irrespective of the value of m, r', and ρ.

A glance will reveal the behavior of the denominator in the Guichard integral on the distant part of the path of integration. Let $x = \xi + i\eta$.

* Williams, loc. cit., § 2.

Then

$$e^{2\pi i(t-x)} - 1 = e^{2\pi i(u-\xi)} \cdot e^{-2\pi(v-\eta)} - 1.$$

It is apparent that along the ray for which $\varphi = \pi/(n+1)$ the quantity just written approaches -1, while along the ray from which $\varphi = -\pi/(n+1)$ it becomes large.

It follows from the considerations given that the integral converges, and will thus furnish a solution of the equation (8).

We shall not inquire into the existence of other solutions or into the behavior of the solution in the infinite part of the plane.

THEOREM 4. *Let p, q, and r satisfy conditions* (1) *and* (2) *of Theorem* 1; *and in addition let $q(x)$ have a pole at infinity. Then it is possible to find a solution of* (8), *such that* (7) *will give a solution of equation* (3), *and therefore of equation* (1).

§ 5.

In the identity (2) we have thus far assumed that $p(x)$ and $q(x)$ were of given form at infinity.

Suppose that $r(x)$ and $p(x)$ are given; then

$$q(x) = \frac{r(x) - p'(x)}{p(x)}.$$

If $p(x)$ is analytic at infinity we may have any of the four cases, depending on the behavior of $r(x)$. If $p(x)$ has a pole at infinity, and $r(x)$ is analytic, or has a pole of lower order than $p(x)$, the equation reduces to Case 2, or possibly Case 1. If $r(x)$ has a pole of the same order as $p(x)$, we have Case 3. If $p(x)$ has a zero at infinity and $r(x)$ does not, the function $q(x)$ will have a pole, and we have Case 4. If $r(x)$ also has a zero at infinity we may have any of the four cases.

Suppose $q(x)$ and $r(x)$ are given. Then

$$p(x) = e^{-\int q(x)dx}\left[\int e^{\int q(x)dx} r(x)dx + C\right].$$

Assume that

$$q(x) = x^{n_1}\left(q_0 + \frac{q_1}{x} + \cdots\right),$$

$$r(x) = x^{n_2}\left(r_0 + \frac{r_1}{x} + \cdots\right), \qquad |x| > R.$$

In order that $p(x)$ have the form of a rational function at infinity we must assume that $n_1 < 0$. If $n_1 = -1$, we must in addition assume that q_0 is an integer (positive or negative) such that $q_0 + n_2 \neq -1$. In all cases not excluded equation (1) will come under a form already treated.

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